Connection between *F*-frames *L* and flat *R(L)*- modules

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Connection between F-frames L and flat $\mathcal{R}(L)$-modules

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Abstract. *In this paper we have shown that, a frame L is an F-frame if and only if every ideal of $\mathcal{R}(L)$ is flat if and only if every submodule of a free $\mathcal{R}(L)$-module is flat.*

1. Introduction

The main intention of this article is to establish the pointfree analogue of the results: (1) X is an F-space if and only if every ideal of $C(X)$ is flat, (2) X is an F-space if and only if every submodule of a free $C(X)$-module is flat; obtained by C. W. Neville [**6**], in 1989. For this purpose, we need the following notations and definitions.

Notation 1. We shall use throughout the paper: (1) the product of two elements $f, g \in \mathcal{R}(L)$ is denoted by fg, (2) for an $\mathcal{R}(L)$-module A, the operation $\mathcal{R}(L) \times A \to A$ is denoted by '.' and the image of an element (r, a) under this operation is denoted by $r.a$ and (3) modules means modules over the ring $\mathcal{R}(L)$.

Definition 1.1. A frame L is called an F-frame, if any two disjoint cozero elements are completely separated i.e. for any $f, g \in \mathcal{R}(L)$ with $cozf \wedge cozg = 0$, there exists $k \in \mathcal{R}(L)$ such that $cozf \wedge cozk = 0$ and $cozg \wedge coz(\mathbf{1} - k) = 0$. For details about these frames see [**1**] and [**4**].

Definition 1.2. A frame L is called a P-frame, if for any $f \in \mathcal{R}(L)$, $cozf \vee (cozf)^* = 1$. For details about these frames see [**1**] and [**3**].

Definition 1.3. A frame L is called basically disconnected, if for any $f \in \mathcal{R}(L)$, $(cozf)^* \vee (cozf)^{**} = 1$. For details about these frames see [**1**].

Definition 1.4. Let $f, g \in \mathcal{R}(L)$. Then we define, $f = 0$ on $supp(g) =\uparrow (cozg)^*$ if $cozf \leq (cozg)^*$ equivalently if $fg = 0$.

Definition 1.5. Let A be a submodule of a free $\mathcal{R}(L)$-module, $a, b \in A$ and $r \in \mathcal{R}(L)$. Then we define, $a = 0$ on $supp(r)$ if $\pi_\alpha(a) = 0$ on $supp(r)$, for each $\alpha \in I$ (since A is a submodule of a free $\mathcal{R}(L)$-module say, $\coprod_{\alpha \in I} \mathcal{R}(L)$, for some index set I, and $\coprod_{\alpha \in I} \mathcal{R}(L)$ is a submodule of $\prod_{\alpha \in I} \mathcal{R}(L)$, we can consider the natural projections π_α's from A into $\mathcal{R}(L)$), and so $r\pi_\alpha(a) = \pi_\alpha(r.a) = 0$, for each $\alpha \in I$, i.e. $r.a = 0$.

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DEFINITION 1.6. A $\mathcal{R}(L)$-module A is called quasi-torsion-free relative to the exact sequence of $\mathcal{R}(L)$-modules

$$0 \longrightarrow K \longrightarrow F \xrightarrow{\beta} A \longrightarrow 0$$

where F is a flat submodule of a free $\mathcal{R}(L)$-module, if the following is true: whenever $r.a = 0$ with $r \in \mathcal{R}(L)$ and $a \in A$, then there exists $b \in F$ and $k \in K$ such that $\beta(b) = a$ and $b = k$ on $supp(r)$ (or equivalently $r.b = r.k$).

For undefined algebraic terminologies, we refer the Rotman's book [**8**] and for general theory of frames and pointfree rings of all real valued continuous functions on them, we refer [**1**], [**2**] and [**7**].

2. Main Results

LEMMA 2.1. *Let $f, g \in \mathcal{R}(L)$ be such that, for any $k \in \mathcal{R}(L)$, $k = 0$ on $supp(f)$ implies $k = 0$ on $supp(g)$. Then $supp(g) \subseteq supp(f)$.*

PROOF. We have to show that $supp(g) \subseteq supp(f)$ equivalently $(coz f)^* \leq (coz g)^*$. If not, then by the complete regularity of L, there exists $k \in \mathcal{R}(L)$ such that $coz k \leq (coz f)^*$ but $coz k \nleq (coz g)^*$, contradicts our hypothesis. □

THEOREM 2.2. *L is an F-frame if and only if every finitely generated ideal of $\mathcal{R}(L)$ is flat.*

PROOF. To prove the theorem, we need a lemma:

LEMMA 2.3. *Let R be a ring. Suppose*

$$0 \longrightarrow K \longrightarrow F \xrightarrow{\beta} B \longrightarrow 0$$

is an exact sequence of R-modules, where F is flat. If B is flat, then $K \cap FI = KI$ for every ideal I of R. Conversely, if $K \cap FI = KI$ for every finitely generated ideal of R, then B is flat.

PROOF. See [**8**, Theorem 3.37]. □

First suppose that L is an F-frame. Let I be a finitely generated ideal of $\mathcal{R}(L)$. Since L is an F-frame, $I = \langle f \rangle$ for some $f \in \mathcal{R}(L)$ (see [**4**, Proposition 3.2]). Now it can be easily checked that

$$0 \longrightarrow K \longrightarrow \mathcal{R}(L) \xrightarrow{\phi} I \longrightarrow 0$$

is an exact sequence of $\mathcal{R}(L)$-modules, where $K = \{k \in \mathcal{R}(L) : kf = 0\}$ and $\phi(g) = fg$, $g \in \mathcal{R}(L)$. Let J be another finitely generated ideal. Then J is also principal and so $J = \langle r \rangle$ for some $r \in \mathcal{R}(L)$. We shall show that $K \cap J = KJ$. Firstly, $KJ \subseteq K \cap J$ always, so we must show that $K \cap J \subseteq KJ$. Let $gr \in K$ with $g \in \mathcal{R}(L)$. Then $grf = 0$ and so $coz(fg) \wedge coz r = 0$. Therefore $coz(fg)$ and $coz r$ are disjoint cozero element of L, also since L is an F-frame, they are completely separated (see [**1**, Proposition 8.4.10]). Hence there exists $h \in \mathcal{R}(L)$ such that $coz(fg) \wedge coz h = 0$ and $coz r \wedge coz(\mathbf{1} - h) = 0$. First equality ensures that $gh \in K$ and second ensures that $gr = ghr \in KJ$. Therefore by the above lemma, we can conclude that I is flat.

Conversely suppose that every finitely generated ideal of $\mathcal{R}(L)$ is flat. To show that L is an F-frame, it is sufficient to show that disjoint cozero elements of L are completely separated (see [**1**, Proposition 8.4.10]). Let $coz f \wedge coz r = 0$ with

$f, r \in \mathcal{R}(L)$. Consider the principal ideals $I = \langle f \rangle$, $J = \langle r \rangle$ and the exact sequence

$$0 \longrightarrow K \longrightarrow \mathcal{R}(L) \xrightarrow{\phi} I \longrightarrow 0$$

with $K = \{k \in \mathcal{R}(L) : kf = 0\}$ and $\phi(g) = fg$, $g \in \mathcal{R}(L)$. Then, since I is flat we have, $K \cap J = KJ$ by the above lemma. Since K is an ideal of $\mathcal{R}(L)$, for any $gr \in K$, $gr = kr$ for some $k \in K$. In particular, being $r \in K$, as $rf = 0$, there exists $k \in K$ such that $r = kr$. So $cozf \wedge cozk = 0$ and $cozr \wedge coz(\mathbf{1} - k) = 0$ and hence $cozf$ and $cozr$ are completely separated by k. □

Also from an well known result (see [**8**, Corollary 3.31]) about flatness is that a module B is flat if every finitely generated submodule is flat. Thus we have:

Corollary 2.4. *L is an F-frame if and only if every ideal of $\mathcal{R}(L)$ is flat.*

From the proof of Theorem 2.2, we also have:

Corollary 2.5. *L is an F-frame if and only if every principal ideal of $\mathcal{R}(L)$ is flat.*

We know from an another well known result (see [**8**, Theorem 4.24]): For any ring R, every R-module is flat if and only if R is von Neumann regular; we obtain the following result:

Theorem 2.6. *L is a P-frame if and only if every $\mathcal{R}(L)$-module is flat.*

Proof. Since we know that, L is a P-frame if and only if $\mathcal{R}(L)$ is a von Neumann regular ring (see [**3**, Proposition 3.9]). □

Lemma 2.7. *If a $\mathcal{R}(L)$-module is quasi-torsion-free relative to one exact sequence, it is quasi-torsion-free relative to every exact sequence.*

Proof. To prove the lemma we need the following lemma:

Lemma 2.8. *Consider the commutative diagram of exact sequences of $\mathcal{R}(L)$-modules*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K_1 & \longrightarrow & F_1 & \xrightarrow{\beta_1} & A & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle i_A} & & \\
0 & \longrightarrow & K_2 & \longrightarrow & F_2 & \xrightarrow[\beta_2]{} & A & \longrightarrow & 0
\end{array}$$

where F_1 and F_2 are flat submodules of free modules. If A is quasi-torsion-free relative to the top exact sequence, it is quasi-torsion-free relative to the bottom exact sequence. If σ is onto and A is quasi-torsion-free relative to the bottom exact sequence, then A is quasi-torsion-free relative to the top exact sequence.

Proof. First suppose that A is quasi-torsion-free relative to the top exact sequence. Let $r.a = 0$ with $r \in \mathcal{R}(L)$ and $a \in A$. Then $a = 0$ on $supp(r) = \uparrow (cozr)^*$ relative to the top exact sequence and so there exists $f_1 \in F_1$ and $k_1 \in K_1$ such that $\beta_1(f_1) = a$ and $f_1 = k_1$ on $supp(r)$ i.e. $r.f_1 = r.k_1$. Take $f_2 = \sigma(f_1)$, $k_2 = \sigma(k_1)$. Then $\beta_2(f_2) = (\beta_2 o \sigma)(f_1) = \beta_1(f_1) = a$ and $f_2 = k_2$ on $supp(r)$ indeed: $r.f_2 = r.\sigma(f_1) = \sigma(r.f_1) = \sigma(r.k_1) = r.\sigma(k_1) = r.k_2$.
Now suppose that σ is onto and A is quasi-torsion-free relative to the bottom exact

sequence. Consider the exact sequence

$$0 \longrightarrow K_3 \longrightarrow F_1 \xrightarrow{\sigma} F_2 \longrightarrow 0$$

and note that $K_3 \cap F_1 I = K_3 I$ for every ideal I of $\mathcal{R}(L)$, by Lemma 2.3. Let $r.a = 0$ with $r \in \mathcal{R}(L)$ and $a \in A$. Then there exists $f_2 \in F_2$ and $k_2 \in K_2$ such that $\beta_2(f_2) = a$ and $r.f_2 = r.k_2$. Since σ is onto, there exists $f_1 \in F_1$ and $k_1 \in K_1$ such that $\sigma(f_1) = f_2$ and $\sigma(k_1) = k_2$. Therefore, $\beta_1(f_1) = (\beta_2 o\sigma)(f_1) = \beta(f_2) = a$ and $\sigma(r.(f_1 - k_1)) = r.f_2 - r.k_2 = 0$ implies $r.(f_1 - k_1) \in K_3 \cap F_1\langle r\rangle = K_3\langle r\rangle$. So, there exists $k_3 \in K_3$ such that $r.(f_1 - k_1) = r.k_3$ i.e. $r.f_1 = r.(k_1 + k_3)$. But, since $\beta_1 = \beta_2 o\sigma$, $K_3 \subseteq K_1$ and hence $(k_1 + k_3) \in K_1$. So A is quasi-torsion-free relative to the top exact sequence. □

Now, if A is quasi-torsion-free with respect to an exact sequence with the middle term free then, since every free module is projective as well as flat, by the first part of the above lemma, A is quasi-torsion-free with respect to every exact sequence. Also, if A is quasi-torsion-free with respect to some exact sequence (as in Definition 1.6) then, since every module is a quotient of a free module F_1 we have an onto map $\sigma : F_1 \to F$ and an exact sequence

$$0 \longrightarrow K_1 \longrightarrow F_1 \xrightarrow{\beta_1} A \longrightarrow 0$$

by letting $\beta_1 = \beta o\sigma$ and $K_1 = \ker(\beta_1)$. Therefore by the second part of the above lemma, we can conclude that A is quasi-torsion-free with respect to this exact sequence with the middle term free. Hence the proof of Lemma 2.7 is complete.

□

THEOREM 2.9. *Let L be an F-frame. Then an $\mathcal{R}(L)$-module is flat if and only if it is quasi-torsion-free.*

PROOF. The '*only if*' part of the above theorem holds for any frame L and the '*if*' part holds for F-frames L.
Let A be a flat $\mathcal{R}(L)$-module and consider the exact sequence (as in Definition 1.6). So by Lemma 2.3, $K \cap FI = KI$, for any ideal I of $\mathcal{R}(L)$. Now, let $r.a = 0$ with $r \in \mathcal{R}(L)$ and $a \in A$. Since β is onto there exists $b \in F$ such that $\beta(b) = a$ and so, $0 = r.a = r.\beta(b) = \beta(r.b)$. Therefore, $r.b \in K \cap F\langle r\rangle = K\langle r\rangle$ implies there exists $c \in K$ such that $r.b = r.c$ i.e. $b = c$ on $supp(r)$ and hence A is quasi-torsion-free.
Conversely suppose that, L is quasi-torsion-free with respect to the exact sequence (as in Definition 1.6). To show A is flat, it is sufficient to show from Lemma 2.3 that, $K \cap FI = KI$ for all principal ideal I of $\mathcal{R}(L)$ as L is an F-frame. Let $I = \langle r\rangle$, $r \in \mathcal{R}(L)$ and $r.b \in K$ with $b \in F$. Then $0 = \beta(r.b) = r.\beta(b)$ and hence by hypothesis, there exists $b_1 \in F$ and $c_1 \in K$ such that $\beta(b) = \beta(b_1)$ (so, $(b - b_1) \in K$) and $b_1 = c_1$ on $supp(r)$ i.e. $r.b_1 = r.c_1$. Now, $r.b = r.(b - b_1 + b_1) = r.(b - b_1) + r.b_1 = r.(b - b_1) + r.c_1 = r.((b - b_1) + c_1)$ implies that $r.b \in KI$ as $((b - b_1) + c_1) \in K$. So, $K \cap FI = KI$. □

Semi-hereditary rings have the property that every finitely generated submodule of a free module is a direct sum of finitely many finitely generated ideals (see [**8**, Theorem 4.13]). Also we have proved in a paper "Concerning finite frames, P-frames and basically disconnected frames", communicated in a journal in January 2013 that, a frame L is basically disconnected if and only if $\mathcal{R}(L)$ is a coherent ring and hence if and only if $\mathcal{R}(L)$ is a semi-hereditary ring, as $\mathcal{R}(L)$ is an uniformly complete, Archimedean, f-algebra with unit (see[**6**], Theorem 3). So we have the following theorem:

THEOREM 2.10. *Let L be basically disconnected. Then every finitely generated submodule of a free $\mathcal{R}(L)$-module is a direct sum of finitely many finitely generated ideals.*

THEOREM 2.11. *L is an F-frame if and only if every finitely generated submodule of a free $\mathcal{R}(L)$-module is flat.*

PROOF. To prove this theorem we need the following lemma:

LEMMA 2.12. *Let L be a frame and A be a submodule of $\coprod_{j\in K}\mathcal{R}(L)$. Let π_j be the canonical projection map onto the j-th coordinate, and assume the ideal $\pi_n(A) = I_n$ is principal with generator $f_n \in \mathcal{R}(L)$. Then the exact sequence*

$$0 \longrightarrow K_n \longrightarrow A \xrightarrow{\pi_n} I_n \longrightarrow 0$$

splits if and only if there exists $a_n \in A$ with $\pi_n(a_n) = f_n$ and $supp(\pi_j(a_n)) \subseteq supp(f_n)$, for all $j \in K$.

PROOF. Assume first that, the exact sequence splits and so there exists a homomorphism $\sigma_n : I_n \to A$ such that $\pi_n o\sigma_n = i_{I_n}$. Take $a_n = \sigma_n(f_n)$, then $a_n \in A$ and $\pi_n(a_n) = (\pi_n o\sigma_n)(f_n) = f_n$. Also since σ_n is a module homomorphism we have for any two $f, g \in \mathcal{R}(L)$, $ff_n = gf_n$ implies $f.a_n = g.a_n$ and so, $f\pi_j(a_n) = g\pi_j(a_n)$ for all $j \in K$. Therefore by Lemma 2.1, $supp(\pi_j(a_n)) \subseteq supp(f_n)$ for all $j \in K$.
Conversely suppose that there exists $a_n \in A$ such that $\pi_n(a_n) = f_n$ and $supp(\pi_j(a_n)) \subseteq supp(f_n)$ for all $j \in K$. Define $\sigma_n : I_n \to A$ by $\sigma_n(ff_n) = f.a_n$, then it is well defined by the given condition, and a module homomorphism. Also for any $f \in \mathcal{R}(L)$, $(\pi_n o\sigma_n)(ff_n) = \pi_n(f.a_n) = f\pi_n(a_n) = ff_n$ implies $\pi_n o\sigma_n = i_{I_n}$ and hence σ_n splits the exact sequence. □

Suppose first that every finitely generated submodule of a free $\mathcal{R}(L)$-module is flat. Since every finitely generated ideal is obviously embeddable in a free module, namely $\mathcal{R}(L)$, L is an F-frame by Theorem 2.2.
Conversely, let L be an F-frame. Let A be a finitely generated submodule of a free module. Then A can be embedded in a finitely generated free module. So without loss of generality $A \subseteq \coprod_1^n \mathcal{R}(L)$. The proof is by induction on n. If $n = 1$ then A is a finitely generated ideal, and so is flat by Theorem 2.2. Suppose $n > 1$ and suppose the theorem has been proved for all finitely generated modules contained in $\coprod_1^{n-1}\mathcal{R}(L)$. Let F_n be the free module $\coprod_1^n \mathcal{R}(L)$. Let π_j and I_n be as in the above lemma. Since L is an F-frame and I_n is finitely generated, $I_n = \langle f_n\rangle$ for some $f_n \in \mathcal{R}(L)$. Consider the homomorphism $\theta : F_n \to F_n$ defined by $\theta(g_1, g_2, ..., g_{n-1}, g_n) = (g_1, g_2, ..., g_{n-1}, g_nf_n)$. Since A is finitely generated and each element of A has a preimage in F_n under θ (let $a = (k_1, k_2, ..., k_n) \in A$, then $k_n = \pi_n(a) = kf_n$, for some $k \in \mathcal{R}(L)$ and so $\theta(k_1, k_2, ..., k) = (k_1, k_2, ..., kf_n = k_n) = a$), there exists a finitely generated submodule B of F_n such that $\theta(B) = A$. Also since $f_n \in \pi_n(A)$, we may assume without loss of generality that there exists $b_n \in B$ such that $\pi_n(b_n) = 1$. Clearly $\pi_n(B) = \langle 1\rangle$. Consider the exact sequence

$$0 \longrightarrow L_n \longrightarrow B \xrightarrow{\pi_n} \langle 1\rangle \longrightarrow 0$$

Trivially the hypotheses of Lemma are satisfied, and so $B = L_n \bigoplus \mathcal{R}(L)$. Clearly L_n is finitely generated (being a homomorphic image of B), and moreover $L_n = \{(g_1, g_2, ..., g_{n-1}, 0) \in B\}$ is embeddable in $\coprod_1^{n-1}\mathcal{R}(L)$, so that L_n is flat by the inductive hypothesis. Thus B is flat, as any free ideal and hence $\mathcal{R}(L)$ is flat and direct sum of two flat modules is flat. Now consider the exact sequence

$$0 \longrightarrow K \longrightarrow B \xrightarrow{\theta} A \longrightarrow 0$$

We shall prove that A is flat by proving that A is quasi-torsion-free relative to this exact sequence. So let, $r.a = 0$ with $r \in \mathcal{R}(L)$ and $a \in A$. We have to show that $a = 0$ on $supp(r)$ with respect to the exact sequence i.e. to find $b \in B$ and $k \in K$ such that $\theta(b) = a$ and $k = b$ on $supp(r)$ i.e. to find $b \in B$ and $k \in K$ such that $\theta(b) = a$ and $r.b = r.k$. Since $\theta(B) = A$, there exists $b = (g_1, g_2, ..., g_n) \in B$ such that $\theta(b) = a$. Also $0 = r.a = r.\theta(g_1, g_2, ..., g_n) = r.(g_1, g_2, ..., g_{n-1}, g_n f_n) = (rg_1, rg_2, ..., rg_{n-1}, rg_n f_n)$ implies $rg_i = 0$ for each $i = 1, 2, .., n-1$ and $rg_n f_n = 0$. Therefore $cozr \wedge (cozg_1 \vee cozg_2 \vee ... \vee cozg_{n-1} \vee cozg_n f_n) = cozr \wedge cozs = 0$ where $s = (g_1^2 + g_2^2 + ... + g_{n-1}^2 + g_n^2 f_n^2) \in \mathcal{R}(L)$. Since L is an F-frame, there exists $t \in \mathcal{R}(L)$ such that $cozr \wedge coz(\mathbf{1} - t) = 0$ and $cozs \wedge cozt = 0$. Therefore $r = rt$ and $st = 0$. Taking $u = t.b$, then $u \in K$ as $\theta(u) = \theta(tg_1, tg_2, ..., tg_n) = (tg_1, tg_2, ..., tg_{n-1}, tg_n f_n) = 0$, since individual components are zero and $st = 0$, and $r.u = r.(t.b) = (rt).b = r.b$. □

Since a module is flat if and only if every finitely generated submodule is flat, we have:

Corollary 2.13. *L is an F-frame if and only if every submodule of a free $\mathcal{R}(L)$-module is flat.*

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