Concerning finite frames, *P*-frames and basically disconnected frames

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Concerning finite frames, P-frames and basically disconnected frames

S. K. Acharyya, G. Bhunia, and Partha Pratim Ghosh

Abstract. *Let $\mathcal{P}$ be an ideal of closed quotients of a completely regular frame L and $\mathcal{R}_{\mathcal{P}}(L)$ be the collection of all functions f in the ring $\mathcal{R}(L)$ whose support belong to $\mathcal{P}$. We have shown that $\mathcal{R}(L)$ is a Noetherian ring if and only if $\mathcal{R}(L)$ is an Artinian ring if and only if L is a finite frame. Using this result we have next shown that if $\mathcal{P}$ is the ideal of all compact closed quotients of L and L is $\mathcal{P}$-continuous then $\mathcal{R}_{\mathcal{P}}(L)$ turns out to be a Noetherian ring if and only if L is finite. Moreover we have established that L is a P-frame if and only if each ideal of $\mathcal{R}(L)$ is of the form $\mathcal{R}_{\mathcal{P}}(L)$ for some choice of $\mathcal{P}$. We have furnished equivalent conditions for $\mathcal{R}_{\mathcal{P}}(L)$ to become a prime ideal, a free ideal and an essential ideal of $\mathcal{R}(L)$ separately in terms of the cozero elements of L. These extend the analogous descriptions for the ideal $\mathcal{R}_{\mathcal{K}}(L)$ of $\mathcal{R}(L)$ as shown in* [**13**]. *Finally we have shown that L is basically disconnected if and only if $\mathcal{R}(L)$ is a coherent ring — analogous to Neville's result for topological space, see* [**17**].

1. Introduction

A principal intention of this paper is to describe each of the three frames L mentioned in the title, in terms of some algebraic properties of the ring $\mathcal{R}(L)$ of all frame maps from the frame $\mathcal{L}(\mathbb{R})$ of reals to a (completely regular) frame L. Most of the results hold good for any frame, although there are some which hold for completely regular frames.

Analogous to the spatial case, it is known that a prime ideal P of $\mathcal{R}(L)$ extends to a unique maximal ideal M (see [**12**, Proposition 5.4]) and the set of all prime ideals that lie between P and M make a chain (see [**11**, Proposition 3.7]). In the present article we have shown, if $P \subsetneq M$, then there exists a strictly ascending chain of upper ideals — prime ideals with an immediate predecessor, of $\mathcal{R}(L)$ in between P and M (see Theorem 2.5), thereby establishing the connection between P-frames (see [**12**] & [**10**], for details) and frames L for which $\mathcal{R}(L)$ is Noetherian. Once we arrive here, using various well known connections between maximal ideals of $\mathcal{R}(L)$, prime elements of the Stone-Čech compactification βL of L and representation of elements of βL in terms of these prime elements we finally characterise in §2 the finite regular frames as precisely those for which their pointfree rings of continuous functions are Noetherian — see Theorem 2.9 & Theorem 2.10. Most of this work depends crucially on the notions developed in [**12**], [**10**] & [**11**]; however,

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we discovered a minor error in the proof Lemma 3.7 of [**10**], which we have repaired — see Example 2.1 and the correct proof thereafter.

We have used a number of characterizations of P-frames as given by [**10**] in order to establish the results in §2. In the present paper we have also provided an alternate characterisation of P-frames (see Theorem 3.11) in terms of ideals of closed quotients of a frame. In fact (ring) ideals of pointfree rings of continuous functions with support as ideals of closed compact quotients were investigated in [**13**]; in the spatial case, rings of continuous functions with compact support or functions which vanish at infinity were investigated in [**1**], [**2**] and the general case in [**3**]. Thus, the pointfree extension was very natural and is precisely dealt in §3. We show, in particular, that for continuous frames the pointfree rings of continuous functions with compact support is a Noetherian ring, if and only if, the underlying set of the frame is finite — see Corollary 3.1, and the principal theorem, Theorem 3.3, which provides much more.

When rings enter to grade spaces, the simplest criterion is to have every prime ideal maximal yielding P-spaces, and then one has the next criteria of demanding finitely generated ideals to be principal yielding F-spaces. The next best is definitely : every finitely generated ideal is finitely presented, and we have a *coherent ring*. Neville (see [**17**]) characterised completely regular topological spaces for which its rings of real valued continuous functions are coherent; in Theorem 4.1 we provide the pointfree analogue of this result, thereby establishing connection with basically disconnected frames.

The results in this paper have used BUT — the Boolean Ultrafilter Theorem, and their instances have been recorded. At some crucial places we could not avoid the use of contrapositive arguments so that all the results are not constructively valid. The constructive version of the results herein is still in progress.

2. Finite frames L versus Noetherian/Artinian rings $\mathcal{R}(L)$.

The following lemma which appears in [**10**, Lemma 3.7] :
Statement of [**10**, Lemma 3.7]: Let Q be an ideal of $\mathcal{R}(L)$. Then :

$$\bigcap \mathcal{M}(Q) = \{\varphi \in \mathcal{R}(L) : r(coz\varphi) \leq \bigvee_{\alpha \in Q} r(coz\alpha)\},$$

where $\mathcal{M}(Q)$ stands for the set of all maximal ideals of $\mathcal{R}(L)$ containing Q. ////

The author in [**10**] argues :
Since I is a maximal element of βL and $I \not\leq \bigvee_{\alpha\in Q} r(coz\alpha)$, it follows that $I \vee \bigvee_{\alpha\in Q} r(coz\alpha) = 1_{\beta L} = L$.

The following counter example substantiates the gap in this argument :

Example 2.1. Choose $L = \mathcal{O}X$, where $X = [0, 1]$; then $\mathcal{R}(L) = C(X)$=the ring of all real valued continuous functions on X. Clearly $L = \beta L$ as L is compact and in this case the join map $j : \beta L \to L$ reduces to the identity map: $L \to L$. Let $I = X - \{\frac{1}{2}\}$ and $Q = \{f \in C(X) : f(\frac{1}{2}) = 0 = f(\frac{1}{3})\}$. Then I is a maximal element of L and Q is an ideal of $\mathcal{R}(L)$. For each element $f \in Q$, $cozf \leq X - \{\frac{1}{2}, \frac{1}{3}\} \lneqq I$. Consequently $I \not\leq \bigvee_{\alpha\in Q} coz\alpha \leq I$, hence $I \vee \bigvee_{\alpha\in Q} coz\alpha = I \neq X = 1_L = 1_{\beta L}$.

Proof of the lemma, using BUT. It was correctly shown in [**10**] :

$$\{\varphi \in \mathcal{R}(L) : r(coz\varphi) \leq \bigvee_{\alpha\in Q} r(coz\alpha)\} \subseteq \bigcap \mathcal{M}(Q).$$

We rectify only the proof of the reversed inequality. So, let $f \in \bigcap \mathcal{M}(Q)$, and if possible let $r(cozf) \not\leq \bigvee_{\alpha\in Q} r(coz\alpha)$.

Then $\bigvee_{\alpha \in Q} r(coz\alpha) \neq 1_{\beta L} = L$. Since any completely regular compact frame is spatial, βL is spatial. Further, from the well known result of Buchi (see [**18**, Proposition 5.3]), every element below the top of a spatial frame is the meet of all the prime elements greater than or equal to it. Hence :

$$(1) \qquad \bigvee_{\alpha \in Q} r(coz\alpha) = \bigwedge_{\gamma \in \Gamma} J_\gamma,$$

where $\{J_\gamma : \gamma \in \Gamma\}$ is a set of prime elements of βL. However, in a completely regular frame prime elements and maximal elements are same (see [**7**]), so that each J_γ is maximal in βL. By our assumption, $r(cozf) \nleq J_\gamma$ for some γ, implying $f \notin M^{J_\gamma}$, a maximal ideal of $\mathcal{R}(L)$ associated with the point $J_\gamma \in \beta L$. On the other hand from (1), $Q \subseteq M^{J_\gamma} \Rightarrow f \in \bigcap \mathcal{M}(Q)$, implying $f \in M^{J_\gamma}$ — a contradiction. $\square$

It is known for any frame L every prime ideal of $\mathcal{R}(L)$ is contained in a unique maximal ideal M (see [**12**, Proposition 5.4]); we now show further, given any non-maximal prime ideal P of $\mathcal{R}(L)$ there exists a strictly increasing chain of prime ideals in between P and M. The development is akin to the development in the spatial case as described in [**15**, Chapter 14].

First, for the existence of n^{th}-roots :

THEOREM 2.1. *Assuming* <u>*BUT*</u>*, for any frame L and for any $n \in \mathbb{N}$, for every positive element f of $\mathcal{R}L$ there exists a positive element g of $\mathcal{R}L$ such that $g^n = f$.*

PROOF. Since $S = \{g \in \mathcal{R}^*(L) : cozg = 1\}$ is a multiplicatively closed subset of $\mathcal{R}^*(L)$ and for each $f \in \mathcal{R}(L)$, $f = \frac{f(1+|f|)^{-1}}{(1+|f|)^{-1}}$, it follows that $\mathcal{R}(L)$ is the ring of fractions of $\mathcal{R}^*(L)$ with respect to S, i.e., $\mathcal{R}(L) = S^{-1}\mathcal{R}^*(L)$.

Since $\mathcal{R}^*(L)$ is isomorphic to $\mathcal{R}(\beta L)$ and βL is a spatial frame, this yields that $\mathcal{R}^*(L)$ is isomorphic to the ring $C(X)$ for some topological space X.

Every positive element of $C(X)$ has a n^{th} root, and hence if f is a positive element of $\mathcal{R}(L)$, then there exist a positive $g \in \mathcal{R}^*(L)$ and a $h \in S$ such that $f = \frac{g}{h}$; but, $g = g_1^n$, $h = h_1^n$ for some $g_1, h_1 \in \mathcal{R}^*(L)$ and of course $h_1 \in S$. Then : $f = (\frac{g_1}{h_1})^n$ with $\frac{g_1}{h_1} \in S^{-1}\mathcal{R}^*(L) = \mathcal{R}(L)$. $\square$

Recall that an ideal I in a commutative lattice ordered ring A with unity is said to be a *l-ideal*, (or, as in [**15**], an *absolutely convex ideal*) if for each $a, b \in A$, $|a| \leq |b|$ and $b \in I$ implies $a \in I$. Such an ideal I in A induces lattice structure on the residue class ring A/I, compatible with its ring structure in the following manner: for any $a \in A$, the residue class $I(a) \geq 0$ if and only if there exists $b \in A$ such that $b \geq 0$ and $a - b \in I$, where $I(a) = \{c \in A : a - c \in I\}$. Furthermore for $a, b \in A$, $I(a \vee b) = I(a) \vee I(b)$, and in particular $I(|a|) = |I(a)|$ (see [**15**, Chapter 5] for details). It was proved by Banaschewski and recorded by Dube in [**11**, Lemma 3.5] that radical ideals of $\mathcal{R}(L)$ are absolutely convex and it is plain that prime ideals are always radical ideals. Hence each prime ideal P in $\mathcal{R}(L)$ is absolutely convex. Therefore the residue class ring $\mathcal{R}(L)/P$ turns out to be a commutative lattice ordered ring with identity, which is further an f-ring if the order is defined as above. Indeed $\mathcal{R}(L)/P$ turns out to be a totally ordered integral domain : for any $\alpha \in \mathcal{R}(L)$, $(|\alpha| + \alpha)(|\alpha| - \alpha) = 4\alpha^+\alpha^- = 0$, where $\alpha^+ = \alpha \vee 0$ and $\alpha^- = (-\alpha) \vee 0$, which implies $(|\alpha| + \alpha) \in P$ or $(|\alpha| - \alpha) \in P$, i.e., $P(\alpha) \leq 0$ or $P(\alpha) \geq 0$, since $|\alpha| \geq 0$. In particular using Theorem 2.1, for any $\alpha \geq 0$ in $\mathcal{R}(L)$ and each $n \in \mathbb{N}$, there is a $\beta \in \mathcal{R}(L)$ such that $\beta^n = \alpha$, and hence $P(\alpha) = (P(\beta))^n$. Thus each positive element in $\mathcal{R}(L)/P$ has an n^{th}-root, which is of course uniquely determined as $\mathcal{R}(L)/P$ is a totally ordered integral domain.

If P is a non maximal prime ideal of $\mathcal{R}(L)$ and M is its unique extension to a maximal ideal, then M/P is the only maximal ideal in the ring $\mathcal{R}(L)/P$. Consequently M/P consists of precisely all the non units of $\mathcal{R}(L)/P$, producing :

Lemma 2.2. *For any frame L, a non-maximal prime ideal P of $\mathcal{R}(L)$ with M as the unique maximal ideal of $\mathcal{R}(L)$ containing it, the maximal ideal M/P of $\mathcal{R}(L)/P$ precisely contains all the non-units of $\mathcal{R}(L)/P$ and :*

$$\alpha, \beta \in \mathcal{R}(L)/P, \alpha \text{ non-unit } \Rightarrow \alpha\beta < 1. \tag{2}$$

Analogous to the spatial case in [**15**, Chapter 14], for any frame L and a prime ideal P of $\mathcal{R}(L)$, an ideal $\mathcal{I} = I/P$ of $\mathcal{R}(L)/P$ is said to be an *upper ideal* if the set of all ideals of $\mathcal{R}(L)/P$ contained in $\mathcal{I}$ has a largest member; the following theorem describes them completely :

Theorem 2.3. *If P is a non maximal prime ideal of the ring $\mathcal{R}(L)$, then the upper ideals of $\mathcal{R}(L)/P$ are precisely $\mathcal{P}^\alpha$ for positive non units α of $\mathcal{R}(L)/P$, where :*

$$\mathcal{P}^\alpha = \bigcap\{\mathcal{Q} : \mathcal{Q} \text{ is a prime ideal of } \mathcal{R}(L)/P \text{ containing } \alpha\} = \bigcup_{n\in\mathbb{N}} \{\beta \in \mathcal{R}(L)/P : |\beta| < \alpha^{\frac{1}{n}}\} \tag{3}$$

Proof. Similar to the spatial case, see [**15**, Theorem 14.6]. □

Theorem 2.4. *An upper ideal of $\mathcal{R}(L)$ is not a z-ideal.*

Proof. Let T be an upper ideal of $\mathcal{R}(L)$ with P its immediate predecessor. Then P is a non maximal prime ideal of $\mathcal{R}(L)$ and T/P is an upper ideal of $\mathcal{R}(L)/P$. By Theorem 2.3, we can write $T/P = \mathcal{P}^\alpha$ for some positive non unit $\alpha = P(a)$ of $\mathcal{R}(L)/P$; obviously, $a \geq 0$ in $\mathcal{R}(L)$. Without any loss of generality, we might choose a such that $0 \leq a \leq 1$ since $P(a \wedge 1) = P(a) \wedge P(1) = P(a)$.

Since $\mathcal{R}(L)$ is uniformly complete (see [**5**, Theorem 4.1.5]), the choice of a ensures the uniform convergence of the infinite series $\sum_{n=1}^{\infty} 2^{-n} a^{\frac{1}{n}}$ to some g in $\mathcal{R}(L)$.

Surely, for each $n \in \mathbb{N}$, $g \geq 2^{-2n} a^{\frac{1}{2n}}$, and an use of [**8**, §6] entails $cozg = \bigvee_{n=1}^{\infty} coz(a^{\frac{1}{n}}) = coza$, which along with Lemma 2.2 imply :

$$P(g) \geq 2^{-2n}\alpha^{\frac{1}{2n}} \geq \alpha^{\frac{1}{2n}}\alpha^{\frac{1}{2n}} = \alpha^{\frac{1}{n}},$$

i.e., for each $n \in \mathbb{N}$, $P(g) \geq \alpha^{\frac{1}{n}}$.

Hence from Theorem 2.3, $P(g) \notin \mathcal{P}^\alpha \Rightarrow g \notin T$ Since $\alpha \in \mathcal{P}^\alpha$, we have $a \in T$, proving T not a z-ideal of $\mathcal{R}(L)$. □

We are now ready to prove the main theorem of this section :

Theorem 2.5 (Main Theorem on Existence of Ascending Chain). *If P is a non maximal prime ideal of $\mathcal{R}(L)$ and M the unique maximal ideal extending P, then there exists a strictly ascending chain of upper ideals of $\mathcal{R}(L)$ which lie between P and M.*

Proof. Since $\psi : I \to I/P$ is an order preserving one-to-one correspondence between the prime ideals I in $\mathcal{R}(L)$ that contain P onto the ideals in $\mathcal{R}(L)/P$, we shall first locate an upper ideal between $\{0\}$ and M/P in $\mathcal{R}(L)/P$. Indeed from Lemma 2.2 any positive element α of M/P turns out to be a positive non unit, and the corresponding prime ideal $\mathcal{P}^\alpha$ is an upper ideal of $\mathcal{R}(L)/P$ by Theorem 2.3, and of course $\{0\} \subsetneqq \mathcal{P}^\alpha \subseteq M/P$.

Hence $\psi^{-1}(\{0\}) \subsetneqq \psi^{-1}(\mathcal{P}^\alpha) \subseteq \psi^{-1}(M/P)$ which means that $P \subsetneqq \psi^{-1}(\mathcal{P}^\alpha) \subseteq M$, with $\psi^{-1}(\mathcal{P}^\alpha)$ an upper ideal in $\mathcal{R}(L)$. Since a maximal ideal in $\mathcal{R}(L)$ is a z-ideal (see [**14**]), Theorem 2.4 yields $P \subsetneqq \psi^{-1}(\mathcal{P}^\alpha) \subsetneqq M$.

Since M is the unique maximal ideal in $\mathcal{R}(L)$ extending the prime ideal $\psi^{-1}(\mathcal{P}^\alpha)$, an use of the Principle of Mathematical Induction would produce a strictly ascending chain of upper ideals in $\mathcal{R}(L)$ between P and M as asserted. □

As an immediate consequence we obtain :

Corollary 2.6. *If $\mathcal{R}(L)$ is a Noetherian ring then L is a P-frame.*

Proof. We prove the contrapositive version of the statement.

If L is not a P-frame, then there exists a non maximal prime ideal P of $\mathcal{R}(L)$ (see [**10**, Proposition 3.9]). By Theorem 2.5 there exists a strictly ascending chain of prime ideals in $\mathcal{R}(L)$ containing P. Hence $\mathcal{R}(L)$ is not Noetherian. □

We shall now characterise the frames L for which $\mathcal{R}(L)$ is a Noetherian ring :

Theorem 2.7. *For any frame L, if $\mathcal{R}(L)$ is a Noetherian ring then the underlying set of L is a finite set.*

Proof. It is enough to prove under the hypothesis the finiteness of the underlying set of βL.

Given the hypothesis, Corollary 2.6 ensures L to be a P-frame; hence each prime ideal in $\mathcal{R}(L)$ is maximal. This entails the Krull-dimension of $\mathcal{R}(L) = 0$.

Since any Noetherian ring with Krull-dimension zero is Artinian there are only finitely many maximal ideals in $\mathcal{R}(L)$ (see [**4**, Proposition 8.3, Theorem 8.5]).

Using the one-to-one correspondence between the maximal ideals of $\mathcal{R}(L)$ and the maximal elements of βL (see [**12**, Lemma 4.15]), it follows that βL has finitely many maximal (and hence prime) elements. Furthermore, using <u>BUT</u>, βL is spatial, and using Buchi's result (see [**18**, Proposition 5.3]) that every element of a spatial frame smaller than the top is a meet of prime elements, it follows that the underlying set of βL is a finite set, proving the assertion. □

Theorem 2.8. *If L is a finite regular frame then $\mathcal{R}(L)$ is a Noetherian ring.*

Proof. Since L is regular, for each $f \in \mathcal{R}(L)$ we can write :

$$coz f = \bigvee \{x \in L : x \prec coz f\},$$

which is essentially a finite join. Therefore $coz f \prec coz f$ and hence $coz f \vee coz f^* = 1$, implying L to be a P-frame.

Consequently each ideal of $\mathcal{R}(L)$ is a z-ideal (see [**10**, Proposition 3.9]); using the one-to-one correspondence between the z-ideals of $\mathcal{R}(L)$ and the ideals of $Coz L$ (see [**12**, page 157]), it follows that there are only finitely many ideals of $\mathcal{R}(L)$, implying $\mathcal{R}(L)$ to be a Noetherian ring. □

Since for any completely regular frame L, $\mathcal{R}^*(L)$ is isomorphic to $\mathcal{R}(\beta L)$ and L is finite when and only when βL is finite, Theorem 2.7 and Theorem 2.8 imply $\mathcal{R}(L)$ is Noetherian if and only if $\mathcal{R}^*(L)$ is Noetherian. Furthermore, a commutative Artinian ring is known to be Noetherian (see [**4**, Theorem 8.5]), leading us to :

Theorem 2.9. *for any completely regular frame L :*

(1) *$\mathcal{R}(L)$ is a Noetherian ring.*
(2) *$\mathcal{R}^*(L)$ is a Noetherian ring.*
(3) *$\mathcal{R}(L)$ is an Artinian ring.*
(4) *$\mathcal{R}^*(L)$ is an Artinian ring.*
(5) *L is a finite set.*

If however L is not completely regular and L^* is the largest completely regular subframe of L, then the two rings $\mathcal{R}(L)$ and $\mathcal{R}(L^*)$ become isomorphic (see [**6**, page (38), last paragraph]). Since each positive element in $\mathcal{R}(L)$ is a square (see [**6**, Proposition 11(3)]) it follows that any isomorphism from $\mathcal{R}(L)$ onto $\mathcal{R}(L^*)$ carries $\mathcal{R}^*(L)$ onto $\mathcal{R}^*(L^*)$. Therefore we can sharpen Theorem 2.9 to obtain :

Theorem 2.10. *for any frame L :*

(1) *$\mathcal{R}(L)$ is a Noetherian ring.*
(2) *$\mathcal{R}^*(L)$ is a Noetherian ring.*
(3) *$\mathcal{R}(L)$ is an Artinian ring.*
(4) *$\mathcal{R}^*(L)$ is an Artinian ring.*
(5) *The largest completely regular subframe L^* of L has its underlying set to be finite.*

3. Functions in $\mathcal{R}(L)$ with ideals of closed quotients as support

For any ideal $\mathcal{P}$ of closed quotients of L, we set

$$\mathcal{R}_{\mathcal{P}}(L) = \{f \in \mathcal{R}(L) :\uparrow (cozf)^* \in \mathcal{P}\}.$$

On taking $\mathcal{P} = \mathcal{K}$, the family of all compact closed quotients of L, we get $\mathcal{R}_{\mathcal{P}}(L) = \mathcal{R}_{\mathcal{K}}(L) = \{f \in \mathcal{R}(L) :\uparrow (cozf)^* \text{ is compact}\}$. The ring $\mathcal{R}_{\mathcal{K}}(L)$ have been recently investigated in [**13**]. We observe the following connection between the cozeros of the functions in $\mathcal{R}_{\mathcal{K}}(L)$ and the continuity of the frame L; recall, a frame L is called continuous if for each $a \in L$, $a = \bigvee\{b \in L : b << a\}$, where $b << a$ means that whenever $S \subseteq L$ is with the condition $a \leq \bigvee S$, the conclusion is $b \leq \bigvee T$ for some finite subset T of S (see [**18**]). In establishing the connection we would require the lemma from [**13**, Corollary 4.14] :

Lemma 3.1. *L is continuous if and only if $\mathcal{R}_{\mathcal{K}}(L)$ is a free ideal of $\mathcal{R}(L)$.*

Theorem 3.2. *L is continuous if and only if $\{cozf : f \in \mathcal{R}_{\mathcal{K}}(L)\}$ generates L.*

Proof. First assume that L is continuous. Then Lemma 3.1 ensures $\bigvee\{cozf : f \in \mathcal{R}_{\mathcal{K}}(L)\} = 1$. Since L is completely regular for each $a \in L$, $a = \bigvee\{cozf_\alpha : f_\alpha \in \mathcal{R}(L), \alpha \in \Lambda\}$ for some index set Λ. Hence $a = \bigvee\{cozf_\alpha : f_\alpha \in \mathcal{R}(L), \alpha \in \Lambda\} \wedge \bigvee\{cozf : f \in \mathcal{R}_{\mathcal{K}}(L)\} = \bigvee\{coz(f_\alpha f) : f \in \mathcal{R}_{\mathcal{K}}(L), f_\alpha \in \mathcal{R}(L), \alpha \in \Lambda\}$. Now since $\mathcal{R}_{\mathcal{K}}(L)$ is an ideal of $\mathcal{R}(L)$ it follows that each $f_\alpha f$ in the last expression is a member of $\mathcal{R}_{\mathcal{K}}(L)$. Hence $\{cozg : g \in \mathcal{R}_{\mathcal{K}}(L)\}$ generates L.

Conversely if $\{cozf : f \in \mathcal{R}_{\mathcal{K}}(L)\}$ generates L, then we can write $1 = \bigvee\{cozf_\beta : f_\beta \in \mathcal{R}_{\mathcal{K}}(L), \beta \in \Gamma\}$ for some index set Γ. This surely implies that $1 = \{cozf : f \in \mathcal{R}_{\mathcal{K}}(L)\}$ i.e. $\mathcal{R}_{\mathcal{K}}(L)$ is a free ideal of $\mathcal{R}(L)$ and Lemma 3.1 completes the proof. □

Motivated from above we define :

Definition 3.1. Given an ideal $\mathcal{P}$ of closed quotients of a frame L we say that L is *$\mathcal{P}$-continuous*, if $\{cozf \; f \in \mathcal{R}_{\mathcal{P}}(L)\}$ generate L.

Thus, $\mathcal{K}$-continuous frames are precisely the continuous frames.

The idea of a frame being compact has its natural generalisation for each infinite cardinal. More precisely, for any regular initial ordinal number ω_α, a frame L is called *finally ω_α-compact*, if for every cover (i.e., a subset $S \subseteq L$ with $\bigvee S = 1$) there exists a subcover of cardinality less than ω_α (i.e., a subset $A \subseteq S$ with cardinality$(A) < \omega_\alpha$ and $\bigvee A = 1$). A culmination of investigations for finally ω_α-compact spatial frames is described in [**19**].

We shall now augment the list of equivalent statements in Theorem 2.10 for $\mathcal{P}$-continuous frames, for suitable specific choices of $\mathcal{P}$:

THEOREM 3.3. *Given any frame L and a regular initial ordinal number ω_α, let $\mathcal{P}_\alpha$ be the ideal of all closed quotients of L which are finally ω_α-compact.*

If L is $\mathcal{P}_\alpha$-continuous, then $\mathcal{R}_{\mathcal{P}_\alpha}(L)$ is a Noetherian ring, if and only if, L is finite.

PROOF. If L is finite, then $\mathcal{R}_{\mathcal{P}_\alpha}(L) = \mathcal{R}(L)$, which is Noetherian by Theorem 2.8.

Conversely, if L is a finally ω_α-compact with infinite underlying set, then again $\mathcal{R}_{\mathcal{P}_\alpha}(L) = \mathcal{R}(L)$, which is not Noetherian by Theorem 2.7.

On the other hand if L is not ω_α-compact, then there is a subset A of L such that $\bigvee A = 1$ but for any subset S of A with $|S| < \omega_\alpha$, $\bigvee S \neq 1$. Let :

$$B = \{\bigvee S : S \text{ is a subset of } A \text{ with cardinality } < \omega_\alpha\}.$$

Then $A \subseteq B$ and hence B is unbounded above with $\bigvee B = 1$.

Choose $b_1 \in B$; since B is unbounded above there exists a $b_0 \in B$ such that $b_0 \not\leq b_1$, and let $b_2 = b_1 \vee b_0 \Rightarrow b_1 \lneqq b_2$.

An use of the Principle of Mathematical Induction then yields a strictly increasing sequence $\{b_n : n \in \omega_0\}$ of elements of B.

If $I_n = \{f \in \mathcal{R}_{\mathcal{P}_\alpha}(L) : coz f \leq b_n\}$, for each $n \in \mathbb{N}$ then I_n is an ideal of $\mathcal{R}_{\mathcal{P}_\alpha}(L)$ with $I_n \subseteq I_{n+1}$. Since L is $\mathcal{P}_\alpha$-continuous, we can write :

$$b_{n+1} = \bigvee\{coz f_\alpha : f_\alpha \in \mathcal{R}_{\mathcal{P}_\alpha}(L), \alpha \in \Lambda\},$$

for some index set Λ. From this we can assert that there is at least one f_α for which $coz f_\alpha \not\leq b_n$, because otherwise it would yield $b_{n+1} \leq b_n$, a contradiction. Therefore $f_\alpha \in I_{n+1}$ but $f_\alpha \notin I_n$. This proves that $I_n \subsetneqq I_{n+1}$ for each $n \in \mathbb{N}$. Hence $\mathcal{R}_{\mathcal{P}_\alpha}(L)$ is not a Noetherian ring. □

COROLLARY 3.4. *For a continuous frame L, $\mathcal{R}_{\mathcal{K}}(L)$ is a Noetherian ring if and only if L is finite.*

COROLLARY 3.5. *For a $\mathcal{L}$-continuous frame L where $\mathcal{L}$ is the ideal of all Lindelöf closed quotients of L, $\mathcal{R}_{\mathcal{L}}(L)$ is a Noetherian ring if and only if L is finite.*

THEOREM 3.6. *$\mathcal{R}_{\mathcal{P}}(L)$ is a proper z-ideal of $\mathcal{R}(L)$, if and only if $L \notin \mathcal{P}$.*

PROOF. Since $\uparrow (coz 1)^* = L$ does not belong to $\mathcal{P}$ if and only if $1 \notin \mathcal{R}_{\mathcal{P}}(L)$, the second part of the first sentence of this theorem is immediate.

If $f, g \in \mathcal{R}_{\mathcal{P}}(L)$ and $h \in \mathcal{R}(L)$ then $coz(f-g) \leq coz f \vee coz g$ implies that $(coz f)^* \wedge (coz g)^* \leq (coz(f-g))^*$, which further implies $\uparrow (coz(f-g))^* \leq \uparrow [(coz f)^* \wedge (coz g)^*] = \uparrow (coz f)^* \vee \uparrow (coz g)^* \in \mathcal{P}$. Hence $\uparrow (coz(f-g))^* \in \mathcal{P}$, so that $(f-g) \in \mathcal{R}_{\mathcal{P}}(L)$.

Similarly, $fh \in \mathcal{R}_{\mathcal{P}}(L)$. □

It is well known in the spatial case for a topological space X that a z-ideal of $C(X)$ is prime if and only if it contains a prime ideal (see [**15**, Theorem 2.9]). We show in this article that the pointfree version of this result is also true. Before proving this, recall for $\alpha, \beta \in \mathcal{R}(L)$, $\alpha \geq \beta$ if and only if $\alpha(r, -) \geq \beta(r, -)$ for each $r \in \mathbb{Q}$ if and only if $\alpha(-, r) \leq \beta(-, r)$ for each $r \in \mathbb{Q}$. In particular for an $\alpha \geq 0$, $coz\alpha = \alpha(0, -)$ (see [**6**, Lemma 4]).

LEMMA 3.7. *for any z-ideal I of $\mathcal{R}(L)$:*

(1) *I is prime.*
(2) *I contains a prime ideal.*

(3) *For all* $g, h \in \mathcal{R}(L)$*, if* $gh = 0$*, then* $g \in I$ *or* $h \in I$.
(4) *Given* $f \in \mathcal{R}(L)$*, there exists a cozero element* $a \in Coz[I] \equiv \{cozg : g \in I\}$ *such that* $f(0, -) \leq a$ *or* $f(-, 0) \leq a$.

Proof. (a)⇒(b) and (b)⇒(c) are trivial.

(c)⇒(d): Let (c) hold and $f \in \mathcal{R}(L)$. Since $(f \vee 0)((-f) \vee 0) = o$, it follows that $(f \vee 0) \in I$ or $((-f) \vee 0) \in I$, say $(f \vee 0) \in I$. Taking $a = coz(f \vee 0)$, we see that $f(0, -) \leq (f \vee 0)(0, -) = coz(f \vee 0) = a$. The other possibility can be tackled analogously.

(d)⇒(a): Let (d) hold and $g, h \in \mathcal{R}(L)$ be such that $gh \in I$. Then for the function $f = |g| - |h| \in \mathcal{R}(L)$, there is a $j \in I$ with $f(0, -) \leq cozj$ say. This yields $coz(|g|) = |g|(0, -) = (f + |h|)(0, -) \leq f(0, -) \vee |h|(0, -) \leq |j|(0, -) \vee |h|(0, -) = coz(|j| + |h|)$. Consequently $cozg = coz(|g|) = coz(|g|) \wedge coz(|j| + |h|) = coz(|gj| + |gh|) = coz(gj) \vee coz(gh) \in Coz[I]$. Since I is a z-ideal it follows that $g \in I$. The possibility $f(-, 0) \leq cozj$ would yield analogously that $h \in I$. We conclude that I is a prime ideal of $\mathcal{R}(L)$. □

The following three theorems are of independent interest telling about the various possibilities regarding the nature of the ideal $\mathcal{R}_{\mathcal{P}}(L)$

Theorem 3.8. *For any ideal* $\mathcal{P}$ *of closed quotients of a frame* L*,* $\mathcal{R}_{\mathcal{P}}(L)$ *is a prime ideal of* $\mathcal{R}(L)$ *if and only if* $L \notin \mathcal{P}$ *and if* $a \wedge b = 0$ *with* $a, b \in CozL$*, then* $\uparrow a^* \in \mathcal{P}$ *or* $\uparrow b^* \in \mathcal{P}$.

Proof. First suppose that $\mathcal{R}_{\mathcal{P}}(L)$ is prime. Then since it is a proper ideal of $\mathcal{R}(L)$ it follows from Theorem 3.6 that $L \notin \mathcal{P}$. If $f, g \in \mathcal{R}(L)$ such that $cozf \wedge cozg = coz(fg) = 0 \Rightarrow fg = 0$, then either $f \in \mathcal{R}_{\mathcal{P}}(L)$ or $g \in \mathcal{R}_{\mathcal{P}}(L)$. Hence $\uparrow (cozf)^* \in \mathcal{P}$ or $\uparrow (cozg)^* \in \mathcal{P}$.

Conversely, choose $f, g \in \mathcal{R}_{\mathcal{P}}(L)$ with $fg = 0$. Then $cozf \wedge cozg = 0$, which yields from hypothesis $\uparrow (cozf)^* \in \mathcal{P}$ or $\uparrow (cozg)^* \in \mathcal{P}$ i.e., $f \in \mathcal{R}_{\mathcal{P}}(L)$ or $g \in \mathcal{R}_{\mathcal{P}}(L)$. Since from Theorem 3.6 $\mathcal{R}_{\mathcal{P}}(L)$ is a z-ideal, it follows from Lemma 3.7 that $\mathcal{R}_{\mathcal{P}}(L)$ is prime. □

Recall that an ideal I of a ring is said to be *essential*, if for any non-zero element a of the ring there exists in I a multiple of a. A simple adaptation of Proposition 4.17 in [**13**] yields :

Theorem 3.9. $\mathcal{R}_{\mathcal{P}}(L)$ *is an essential ideal of* $\mathcal{R}(L)$ *if and only if* L *is almost* $\mathcal{P}$*-continuous.*

Theorem 3.10. *For any ideal* $\mathcal{P}$ *of closed quotients of a completely regular frame* L*,* $\mathcal{R}_{\mathcal{P}}(L)$ *is a free ideal of* $\mathcal{R}(L)$ *if and only if* L *is* $\mathcal{P}$*-continuous.*

Proof. Assume that $\mathcal{R}_{\mathcal{P}}(L)$ is a free ideal, i.e., $\bigvee\{cozf : f \in \mathcal{R}_{\mathcal{P}}(L)\} = 1$. The complete regularity of L entails $CozL$ generates L, and hence for any $a \in L$, there exists an index set Λ, such that $a = \bigvee\{cozg_\alpha : g_\alpha \in \mathcal{R}(L), \alpha \in \Lambda\}$. Consequently :

$$a = \bigvee\{cozg_\alpha : g_\alpha \in \mathcal{R}(L), \alpha \in \Lambda\} \wedge \bigvee\{cozf : f \in \mathcal{R}_{\mathcal{P}}(L)\}$$
$$= \bigvee\{coz(g_\alpha f) : g_\alpha \in \mathcal{R}(L), f \in \mathcal{R}_{\mathcal{P}}(L), \alpha \in \Lambda\},$$

and we note that each $g_\alpha f \in \mathcal{R}_{\mathcal{P}}(L)$. Thus $\{cozf : f \in \mathcal{R}_{\mathcal{P}}(L)\}$ generates L, implying L to be $\mathcal{P}$-continuous.

The other part of the theorem is trivial. □

The next result describes P-frames L via ideals of closed quotients of L.

THEOREM 3.11. *L is a P-frame if and only if each ideal of $\mathcal{R}(L)$ is of the form $\mathcal{R}_{\mathcal{P}}(L)$ for some ideal $\mathcal{P}$ of closed quotients of L.*

PROOF. If each ideal of $\mathcal{R}(L)$ is of the form $\mathcal{R}_{\mathcal{P}}(L)$ which by Theorem 3.6 is a z-ideal, then [**10**, Proposition 3.9] implies L to be a P-frame.

Conversely assume that L is a P-frame and I an ideal of $\mathcal{R}(L)$. Set $\mathcal{P} = \{\uparrow a : a \in L$ and there exists $f \in I$ such that $\uparrow a \leq \uparrow (cozf)^*\}$; which is evidently an ideal of closed quotients of L.

We shall show : $I = \mathcal{R}_{\mathcal{P}}(L)$.

It is plain that $I \subseteq \mathcal{R}_{\mathcal{P}}(L)$; for the reverse inclusion, choose any $f \in \mathcal{R}_{\mathcal{P}}(L)$. Then $\uparrow (cozf)^* \in \mathcal{P}$, and hence there exists $g \in I$ such that $\uparrow (cozf)^* \leq \uparrow (cozg)^*$. Consequently $(cozg)^* \leq (cozf)^* \Rightarrow cozf = (cozf)^{**} \leq (cozg)^{**} = cozg$, where we use the fact that for the P-frame L, for any $h \in \mathcal{R}(L)$, $cozh = (cozh)^{**}$, a relation easily verifiable. This yields, $cozf = cozf \wedge cozg = coz(fg)$; but $g \in I$ implies that $fg \in I$, moreover I is a z-ideal of $\mathcal{R}(L)$ because L is a P-frame. Hence it follows that $f \in I$. This proves that $\mathcal{R}_{\mathcal{P}}(L) \subseteq I$. □

4. Basically disconnected frames L versus coherent rings $\mathcal{R}(L)$.

This section contains just a single proposition.

THEOREM 4.1. *for any completely regular frame L :*

(1) *L is basically disconnected.*
(2) *$\mathcal{R}(L)$ is a coherent ring.*
(3) *$\mathcal{R}^*(L)$ is a coherent ring.*

PROOF. [**9**, Proposition 2] tells that L is a basically disconnected frame if and only if $\mathcal{R}(L)$ is Dedekind σ-complete which means that for any countable subset $\{f_n : n \in \omega_0\}$ of $\mathcal{R}(L)$, $\bigvee\{f_n : n \in \omega_0\} \in \mathcal{R}(L)$. However, [**16**, Theorem 3] ensures $\mathcal{R}(L)$ is Dedekind σ-complete, if and only if, $\mathcal{R}(L)$ is a coherent ring; hence the equivalence of the first two statements follow.

On the other hand, from [**5**, Proposition 8.4.5], L is basically disconnected if and only if βL is basically disconnected. Since $\mathcal{R}^*(L)$ is isomorphic to $\mathcal{R}(\beta L)$, it follows that $\mathcal{R}(L)$ is a coherent ring precisely when $\mathcal{R}^*(L)$ is a coherent ring, thereby completing the proof. □

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